[2023-01-26.10:42 Thu > The Forward-Path Ratio R in Special Relativity

The Forward-Path Ratio *R* in Special Relativity

DRAFT PAPER

Abstract: A moving object undergoes an internal race with light. The object's motion causes a forward-moving light pulse to travel farther as it races the object's motion in the same forward direction. Conversely, a backward-moving light pulse travels a shorter distance before colliding with the oncoming object's trailing side. The effects are reciprocal, so one ratio R of the forward path distance for the moving case over the same distance for the resting object case captures both effects as R and $1/R$. This ratio captures the fundamental geometry of special relativity transformations better than the Lorentz factor, which turns out to be an average of R and $1/R$. Other examples where R shows up in special relativity include the particle physics metric *rapidity*, which is the natural log of R, and the astrophysics *Doppler factor*, which is precisely R. This paper describes the geometry of R, discusses its uses, and provides a table of interconversions with other special relativity representations of velocity.

Six Velocity Metrics

What is the best way to represent velocity in special relativity? Figure 1 gives examples of six possible metrics in the context of a Minkowski light cone figure.

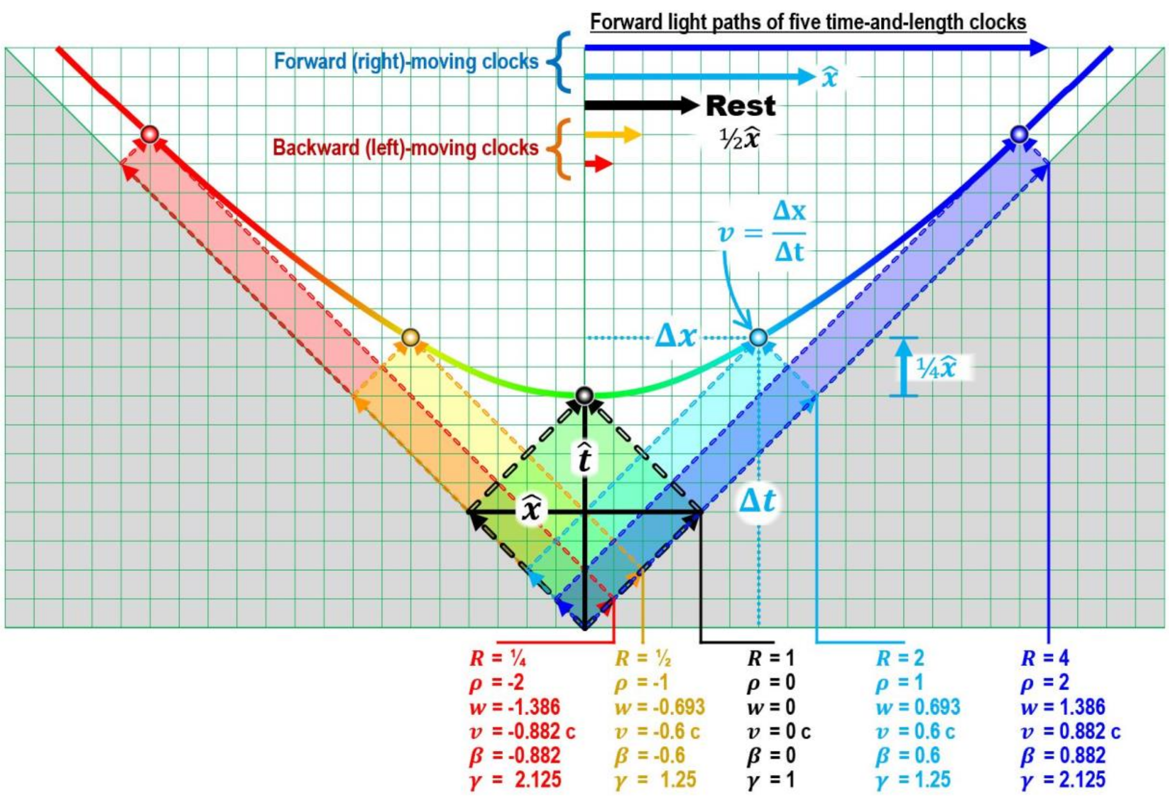

Figure 1. *Six special relativity velocity metrics*

Below the figure are the six metrics for $\pm 0.882\,c$, $\pm 0.6\,c$, and rest. Each rectangle is one tick of a *javelin clock* that defines moving length and time units in terms of light paths.

Javelin Clocks

Figure 2 shows the design of the javelin clocks that create the Figure 1 rectangles.

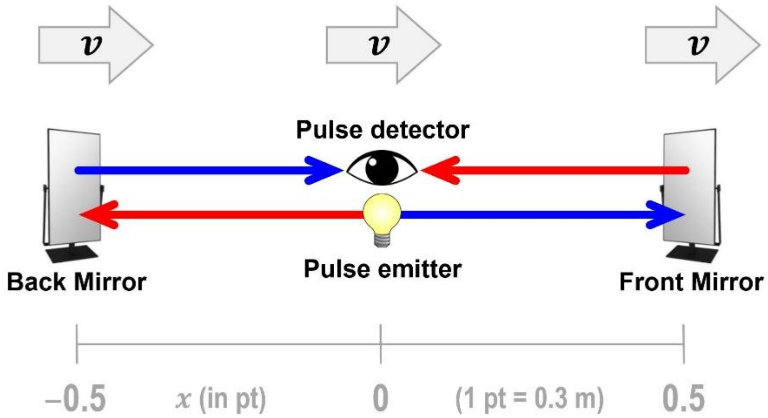

Figure 2. *Design of a Javelin Clock*

A javelin clock consists of a left (backward) mirror, a right (forward) mirror, and an emitter-detector at the origin, midway between the mirrors. All three components move together, javelin-style, along the x axis, with positive velocity on the right.

Figure 3 shows the progress of light pulses over one cycle of a javelin clock at rest.

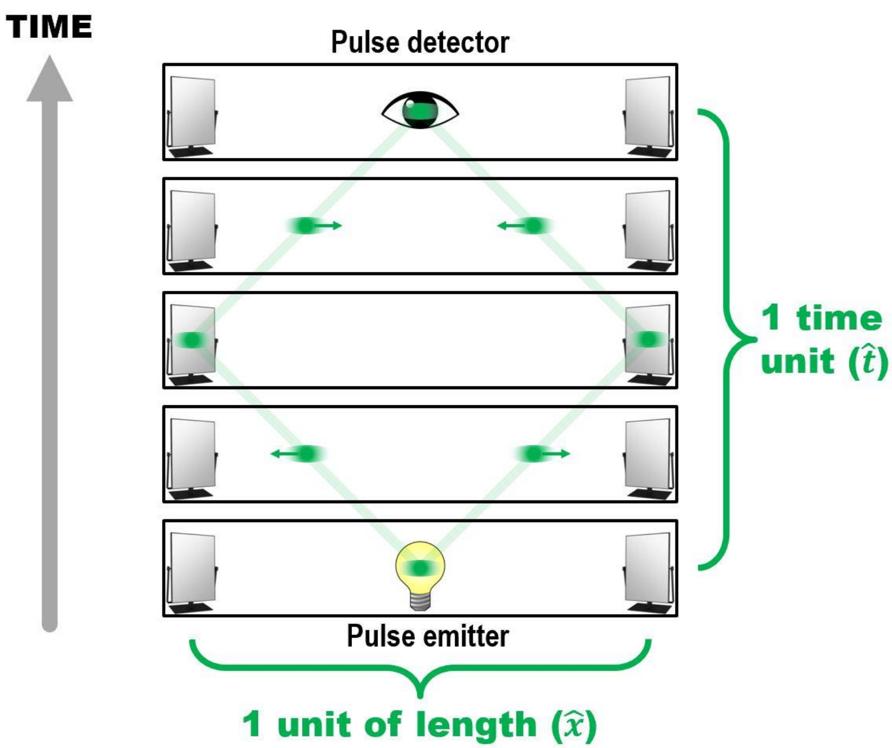

Figure 3. *The light pulse paths traced by one cycle of a javelin clock at rest*

A javelin clock cycle begins with the emitter sending synchronized light pulses forward and backward. The left and right mirrors then return these pulses to the detector to complete the cycle. The calibrated distance between the mirrors then defines one unit of length, $\hat{x}$, and the time required for the round trips defines the time unit $\hat{t}$. Basing $\hat{x}$ and $\hat{t}$ on the same light paths ensures the preservation of the relationship between the space and time metrics when the clock is moving.

As in all Minkowski light cone diagrams, Figure 1 and Figure 3 scale $\hat{x}$ and $\hat{t}$ so that $\hat{t} = \hat{x}/c$. This ensures that light always moves to the left or right at $45°$ angles. Since the left and right light pulses begin and end at the same points in space and time, they necessarily trace out rectangles tilted at $45°$.

Figure 4 gives a slightly more abstract view of pulse tracking for a $0.3\ m$ long javelin clock at rest and moving to the right at $0.6\ c$. The moving case demonstrates the often peculiar impacts of special relativity on the length, time, and shape of the clock cycle.

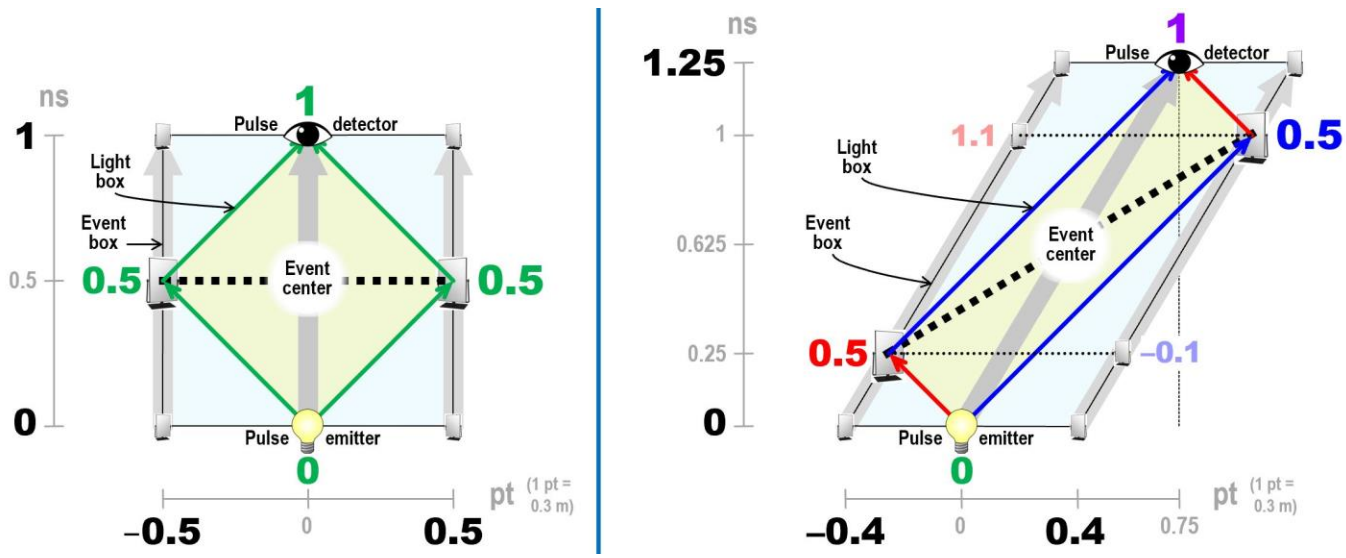

Figure 4. *A 0.3-meter javelin clock at rest and moving at 0.6 c*

Lorentz Areas and Lorentz Area Invariance

An important property of the light-path rectangles in Figure 4 is that the areas of the two rectangles, shown in yellow, remain the same regardless of the velocity of the clock. The source of this area invariance is that the rectangles ultimately are products of the clock's Lorentz-dilated time, $\Delta t \cdot \gamma$, and Lorentz-compressed length, $\Delta x/\gamma$. The reciprocal Lorentz factors cancel out to make this *Lorentz area* invariant across all clock velocities.

Lorentz area conservation is even more apparent in Figure 1, where it forces the velocity rectangles to grow increasingly thin and elongated at high velocities. Only the rest-frame clock forms a perfect square. A square Lorentz area defines the rest frame in figures with multiple velocities.

Finally, the fact that canceling Lorentz factors produces a relativistically invariant quantity suggests that applying it individually to *either* space or time may not be the best way to capture the geometry of such figures. What, exactly, is the Lorentz factor in such figures?

The Lorentz Factor as a Composite Metric

The metric most often used to describe how motion affects the shape and dynamics of a fast-moving object is the Lorentz factor γ, since the length x of an object in the direction of motion contracts by x/γ, while its seconds expand (dilate) by $t\gamma$. Given $\beta = v/c$:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{c}{\sqrt{c^2 - v^2}} = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{(1+\beta)(1-\beta)}}$$

A less obvious way to express γ is as the *average* of a number R and its reciprocal:

$$\gamma = \frac{\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}}}{2} = \frac{\sqrt{\frac{1+\beta}{1-\beta}} + 1\Big/\sqrt{\frac{1+\beta}{1-\beta}}}{2} = \frac{\boldsymbol{R} + \frac{\boldsymbol{1}}{\boldsymbol{R}}}{\boldsymbol{2}} \qquad \text{... where } R = \sqrt{\frac{1+\beta}{1-\beta}}$$

This non-obvious equivalence merits verification:

$$\frac{R + 1/R}{2} = \frac{R^2+1}{2R} = \frac{\frac{1+\beta}{1-\beta} + 1}{2\sqrt{\frac{1+\beta}{1-\beta}}} = \frac{\frac{1+\beta+1-\beta}{1-\beta}}{2\sqrt{\frac{(1+\beta)(1-\beta)}{(1-\beta)^2}}} = \frac{\frac{2}{1-\beta}}{\frac{2}{1-\beta}\cdot\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

So what is R? What does it mean physically, and how does it break the Lorentz factor into an average of two versions of R? The latter is especially intriguing since it suggests γ is a composite metric of R, making the R the more fundamental unit.

The physical explanation of R is not complicated: It is the ratio of how far a forward light pulse travels when an object is moving compared to how far it travels when the object is at rest. Even backward-moving clocks have this forward-moving component, though in that case, the distance is shortened rather than lengthened. The five bars at the top of Figure 1 show examples of forward path lengths. The calibrating rest length is ½$\hat{x}$, or half of the clock's total length.

The Role of R in Relativity Metrics

The fact that Lorentz areas remain invariant for all velocities provides a different way of interpreting R, which sheds some light on its importance. While Lorentz areas remain invariant regardless of the observer, the *shape* of a Lorentz area can vary wildly. Square Lorentz areas indicate objects at rest relative to the observer, while thin, highly distorted ones indicate objects moving close to lightspeed. The R value is the most direct way of indicating this distortion effect by focusing on the ratio of stretching of the Lorentz area in the forward direction. Since the Lorentz area is invariant, setting this forward-path ratio to R also sets the backward-path ratio to $1/R$. The Lorentz factor captures this relationship in a more distorted form by averaging these two equally important ratios.

The relationship of R to velocity means shows up not just in the Lorentz factor but in other relativistic variants of velocity. Table 1 lists ten such velocity-related factors and provides equations for converting each into one of the other versions.

	Velocity $\boldsymbol{v} =$	*Unitless Velocity* $\boldsymbol{\beta} =$	*Length Ratio of Forward Light Paths* $\boldsymbol{R} =$	*Rapidity* $\boldsymbol{w} =$	*Binary Rapidity* $\boldsymbol{\rho} =$	*Lorentz Factor* $\boldsymbol{\gamma} =$	*Diagonal Factor* $\boldsymbol{D} =$	*Age Gradient* $\boldsymbol{\alpha} =$	*Traveler's Gradient* $\boldsymbol{\alpha_+} =$	*In-Frame Gradient* $\boldsymbol{\alpha'} =$
Best →	v	$\frac{v}{c}$	$\sqrt{\frac{1+\beta}{1-\beta}}$	$\ln R$	$\log_2 R$	$\frac{R+R^{-1}}{2}$	$\sqrt{2\gamma^2-1}$	$-\frac{\beta\gamma}{c}$	$-\alpha$	$-\frac{v}{c^2}$
Given ↓ $\boldsymbol{v}$	v	$\frac{v}{c}$	$\sqrt{\frac{c+v}{c-v}}$	$\ln\sqrt{\frac{c+v}{c-v}}$	$\log_2\sqrt{\frac{c+v}{c-v}}$	$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$	$\sqrt{\frac{c^2+v^2}{c^2-v^2}}$	$-\frac{v}{c\sqrt{c^2-v^2}}$	$\frac{v}{c\sqrt{c^2-v^2}}$	$-\frac{v}{c^2}$
$\boldsymbol{\beta}$	$c\beta$	β	$\sqrt{\frac{1+\beta}{1-\beta}}$	$\ln\sqrt{\frac{1+\beta}{1-\beta}}$	$\log_2\sqrt{\frac{1+\beta}{1-\beta}}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\sqrt{\frac{1+\beta^2}{1-\beta^2}}$	$-\frac{\beta}{c\sqrt{1-\beta^2}}$	$\frac{\beta}{c\sqrt{1-\beta^2}}$	$-\frac{\beta}{c}$
$\boldsymbol{R}$	$c\frac{R-R^{-1}}{R+R^{-1}}$	$\frac{R-R^{-1}}{R+R^{-1}}$	R	$\ln R$	$\log_2 R$	$\frac{R+R^{-1}}{2}$	$\sqrt{\frac{R^2+R^{-2}}{2}}$	$-\frac{R-R^{-1}}{2c}$	$\frac{R-R^{-1}}{2c}$	$-\frac{R-R^{-1}}{2c}$
$\boldsymbol{w}$	$c\frac{e^w-e^{-w}}{e^w+e^{-w}}$	$\frac{e^w-e^{-w}}{e^w+e^{-w}}$	e^w	w	$\frac{w}{\ln 2}$	$\frac{e^w+e^{-w}}{2}$	$\sqrt{\frac{e^{2w}+e^{-2w}}{2}}$	$-\frac{e^w-e^{-w}}{2c}$	$\frac{e^w-e^{-w}}{2c}$	$-\frac{e^w-e^{-w}}{2c}$
$\boldsymbol{\rho}$	$c\frac{2^\rho-2^{-\rho}}{2^\rho+2^{-\rho}}$	$\frac{2^\rho-2^{-\rho}}{2^\rho+2^{-\rho}}$	2^ρ	$\rho\ln 2$	ρ	$\frac{2^\rho+2^{-\rho}}{2}$	$\sqrt{\frac{2^{2\rho}+2^{-2\rho}}{2}}$	$-\frac{2^\rho-2^{-\rho}}{2c}$	$\frac{2^\rho-2^{-\rho}}{2c}$	$-\frac{2^\rho-2^{-\rho}}{2c}$
$\boldsymbol{\gamma}$	$c\sqrt{1-\frac{1}{\gamma^2}}$	$\sqrt{1-\frac{1}{\gamma^2}}$	$\sqrt{\frac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}}}$	$\ln\sqrt{\frac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}}}$	$\log_2\sqrt{\frac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}}}$	γ	$\sqrt{2\gamma^2-1}$	$-\frac{\sqrt{\gamma^2-1}}{c}$	$\frac{\sqrt{\gamma^2-1}}{c}$	$-\frac{\sqrt{\gamma^2-1}}{c}$
$\boldsymbol{D}$	$c\sqrt{\frac{D^2-1}{D^2+1}}$	$\sqrt{\frac{D^2-1}{D^2+1}}$	$\sqrt{\frac{1+\sqrt{\frac{D^2-1}{D^2+1}}}{1-\sqrt{\frac{D^2-1}{D^2+1}}}}$	$\ln\sqrt{\frac{1+\sqrt{\frac{D^2-1}{D^2+1}}}{1-\sqrt{\frac{D^2-1}{D^2+1}}}}$	$\log_2\sqrt{\frac{1+\sqrt{\frac{D^2-1}{D^2+1}}}{1-\sqrt{\frac{D^2-1}{D^2+1}}}}$	$\sqrt{\frac{D^2+1}{2}}$	D	$-\sqrt{\frac{D^2-1}{2c^2}}$	$\sqrt{\frac{D^2-1}{2c^2}}$	$-\sqrt{\frac{D^2-1}{2c^2}}$
$\boldsymbol{\alpha}$	$-\frac{\alpha c^2}{\sqrt{1+\alpha^2c^2}}$	$-\frac{\alpha c}{\sqrt{1+\alpha^2c^2}}$	$\sqrt{\frac{\sqrt{1+\frac{1}{\alpha^2c^2}}+1}{\sqrt{1+\frac{1}{\alpha^2c^2}}-1}}$	$\ln\sqrt{\frac{\sqrt{1+\frac{1}{\alpha^2c^2}}+1}{\sqrt{1+\frac{1}{\alpha^2c^2}}-1}}$	$\log_2\sqrt{\frac{\sqrt{1+\frac{1}{\alpha^2c^2}}+1}{\sqrt{1+\frac{1}{\alpha^2c^2}}-1}}$	$\sqrt{1+\alpha^2c^2}$	$\sqrt{1+2\alpha^2c^2}$	α	$-\alpha$	$\frac{\alpha}{\sqrt{1+\alpha^2c^2}}$
$\boldsymbol{\alpha_+}$	$\frac{\alpha_+c^2}{\sqrt{1+\alpha_+^2c^2}}$	$\frac{\alpha_+c}{\sqrt{1+\alpha_+^2c^2}}$	$\sqrt{\frac{\sqrt{1+\frac{1}{\alpha_+^2c^2}}+1}{\sqrt{1+\frac{1}{\alpha_+^2c^2}}-1}}$	$\ln\sqrt{\frac{\sqrt{1+\frac{1}{\alpha_+^2c^2}}+1}{\sqrt{1+\frac{1}{\alpha_+^2c^2}}-1}}$	$\log_2\sqrt{\frac{\sqrt{1+\frac{1}{\alpha_+^2c^2}}+1}{\sqrt{1+\frac{1}{\alpha_+^2c^2}}-1}}$	$\sqrt{1+\alpha_+^2c^2}$	$\sqrt{1+2\alpha_+^2c^2}$	$-\alpha_+$	α_+	$-\frac{\alpha_+}{\sqrt{1+\alpha_+^2c^2}}$
$\boldsymbol{\alpha'}$	$-\alpha'c^2$	$-\alpha'c$	$\sqrt{\frac{1-\alpha'c}{1+\alpha'c}}$	$\ln\sqrt{\frac{1-\alpha'c}{1+\alpha'c}}$	$\log_2\sqrt{\frac{1-\alpha'c}{1+\alpha'c}}$	$\frac{1}{\sqrt{1+\alpha'^2c^2}}$	$\sqrt{\frac{1+\alpha'^2c^2}{1-\alpha'^2c^2}}$	$\frac{\alpha'}{\sqrt{1-\alpha'^2}}$	$-\frac{\alpha'}{\sqrt{1-\alpha'^2}}$	α'

Table 1. *Ten special relativity factors and their interconversion formulas*

Velocity v and unitless velocity β are standard and the most common experimental starting points for determining an object's velocity state, at least for velocities significantly less than the speed of light. The ratio R is not only an essential geometric number but is also identical to a value called the relativistic Doppler factor. Particle physicists typically use rapidity w (or sometimes y) as a better way of representing velocities very close to lightspeed. Interestingly, R provides an exceptionally straightforward way to express rapidity since rapidity is just its natural log. Since any log provides this representational benefit, *binary rapidity* ρ, which is rapidity divided by the square root of 2, provides a more geometrically intuitive version of rapidity for use in figures.

The Diagonal factor D describes the increase in the diagonal length of the moving-frame light path rectangles and gives the moving-frame Minkowski lengths of $\widehat{x'}$ and $\widehat{t'}$. When viewed from the rest frame the $\widehat{x'}$ and $\widehat{t'}$ diagonals amount to not much more than two additional velocity paths. This lack of unity makes them less helpful to calculations than one might expect but does make the point that *all* units of length and time, including those of the observer, become time-sequenced dynamic effects in other frames.

The three *age gradients* support three viewpoints for finding the slope in time labels that necessarily emerge during Lorentz compression. Given their graphical appearance in every derivation of Lorentz contraction, the absence of these equations in textbooks is baffling.

To finish, Table 2 repeats Table 1 using copy-and-substitutable Google equations.

	Velocity v V =	*Unitless Velocity* β B =	*Length Ratio of Forward Light Paths* R R =	*Rapidity* w W =	*Binary Rapidity* ρ P =	*Lorentz Factor* γ Y =	*Diagonal Factor* D D =	*Age Gradient* α A =	*Traveler's Gradient* α_+ F =	*In-Frame Gradient* α' J =
Best→	(V)	(V)/c	√((1+(B))/(1-(B)))	ln((R))	log_2((R))	((R)+(R)^(-1))/2	√(2((Y)^2)–1)	-(B)(Y)/c	-(A)	-(V)/(c^2)
Given↓										
v: V	(V)	(V)/c	√((c+(V))/(c-(V)))	ln(√((c+(V))/(c-(V))))	log_2(√((c+(V))/(c-(V))))	1/√(1-((V)^2)/(c^2))	√(((c^2)+((V)^2))/((c^2)-((V)^2)))	-(V)/(c√((c^2)-((V)^2)))	(V)/(c√((c^2)-((V)^2)))	-(V)/(c^2)
β: B	c(B)	(B)	√((1+(B))/(1-(B)))	ln(√((1+(B))/(1-(B))))	log_2(√((1+(B))/(1-(B))))	1/√(1-(B)^2)	√((1+(B)^2)/(1-(B)^2))	-(B)/(c√(1-(B)^2))	(B)/(c√(1-(B)^2))	-(B)/c
D: R	c((R)-(R)^(-1))/((R)+(R)^(-1))	((R)-(R)^(-1))/((R)+(R)^(-1))	(R)	ln((R))	log_2((R))	((R)+(R)^(-1))/2	√(((R)^2+(R)^(-2))/2)	-((R)-(R)^(-1))/2c	((R)-(R)^(-1))/2c	-((R)-(R)^(-1))/2c
w: W	c((e^(W))-e^(-(W)))/((e^(W))+e^(-(W)))	((e^(W))-e^(-(W)))/((e^(W))+e^(-(W)))	(e^(W))	(W)	(W)/ln(2)	((e^(W))+e^(-(W)))/2	√(((e^(2(W)))+(e^(-2(W))))/2)	-((e^(W))-e^(-(W)))/2c	((e^(W))-e^(-(W)))/2c	-((e^(W))-e^(-(W)))/2c
ρ: P	c((2^(P))-(2^(-(P))))/((2^(P))+(2^(-(P))))	((2^(P))-(2^(-(P))))/((2^(P))+(2^(-(P))))	(2^(P))	(P)ln(2)	(P)	((2^(P))+(2^(-(P))))/2	√(((2^(2(P)))+(2^(-2(P))))/2)	-((2^(P))-(2^(-(P))))/2c	((2^(P))-(2^(-(P))))/2c	-((2^(P))-(2^(-(P))))/2c
γ: Y	c√(1-1/((Y)^2))	√(1-1/((Y)^2))	√(((Y)+√(((Y)^2)-1))/((Y)-√(((Y)^2)-1)))	ln(√(((Y)+√(((Y)^2)–1))/((Y)-√(((Y)^2)-1))))	log_2(√(((Y)+√(((Y)^2)-1))/((Y)-√(((Y)^2)-1))))	(Y)	√(2((Y)^2)–1)	-√(((Y)^2)-1)/c	√(((Y)^2)-1)/c	-√(((Y)^2)-1)/c
D: D	c√((((D)^2)-1)/(((D)^2)+1))	√((((D)^2)-1)/(((D)^2)+1))	√((1+√((((D)^2)-1)/(((D)^2)+1)))/(1-√((((D)^2)-1)/(((D)^2)+1))))	ln(√((1+√((((D)^2)-1)/(((D)^2)+1)))/(1-√((((D)^2)-1)/(((D)^2)+1)))))	log_2(√((1+√((((D)^2)-1)/(((D)^2)+1)))/(1-√((((D)^2)-1)/(((D)^2)+1)))))	√((((D)^2)+1)/2)	(D)	-√((((D)^2)-1)/(2(c^2)))	√((((D)^2)-1)/(2(c^2)))	-√((((D)^2)-1)/(2(c^2)))
α: A	-((A)(c^2))/√(1+((A)^2)(c^2))	-(A)c/√(1+((A)^2)(c^2))	√((√(1+1/(((A)^2)(c^2)))+1)/(√(1+1/(((A)^2)(c^2)))-1))	ln(√((√(1+1/(((A)^2)(c^2)))+1)/(√(1+1/(((A)^2)(c^2)))-1)))	log_2(√((√(1+1/(((A)^2)(c^2)))+1)/(√(1+1/(((A)^2)(c^2)))-1)))	√(1+((A)^2)(c^2))	√(1+2((A)^2)(c^2))	(A)	-(A)	(A)/√(1+((A)^2)(c^2))
α_+: F	((F)(c^2))/√(1+((F)^2)(c^2))	((F)c)/√(1+((F)^2)(c^2))	√((√(1+1/(((F)^2)(c^2)))+1)/(√(1+1/(((F)^2)(c^2)))-1))	ln(√((√(1+1/(((F)^2)(c^2)))+1)/(√(1+1/(((F)^2)(c^2)))-1)))	log_2(√((√(1+1/(((F)^2)(c^2)))+1)/(√(1+1/(((F)^2)(c^2)))-1)))	√(1+((F)^2)(c^2))	√(1+2((F)^2)(c^2))	-(F)	(F)	-(F)/√(1+((F)^2)(c^2))
α': J	-(J)(c^2)	-(J)c	√((1-(J)c)/(1+(J)c))	ln(√((1-(J)c)/(1+(J)c)))	log_2(√((1-(J)c)/(1+(J)c)))	1/√(1+(((J)^2)(c^2))	√((1+(((J)^2))(c^2))/(1-(((J)^2))(c^2)))	(J)/√(1-((J)^2))	-(J)/√(1-((J)^2))	(J)

Table 1. *Copy-and-substitute Google equations for interconverting special relativity factors*