The Yang-Mills Problem (Millennium Problem II-3)

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(A Relevant Earlier Atiyah Comment from Problem II-1, the Poincaré Conjecture)

[17:55] This [Simon Donaldson] four-dimensional theory was a very unexpected development — I would say the most unexpected development of recent times. And, again, the influence all came from physics. The invariants come from the use of the Yang-Mills equations and, subsequently, the Seiberg-Witten equations, which I may say more about later. This was a very spectacular result.
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Problem II-3: Yang-Mills Theory

[29:25] Well, now I'm going to move on from Problem 2 to Problem 3. I'm halfway through!

[29:30] Now, the first two problems were problems of pure mathematics. They are problems of geometry, algebra, analysis, and differential equations as background, as well. Problems 3 and 4 have to do with physics. Problem 3 has to do with the Yang-Mills theory. So, let me try to explain what this problem is, and where we are.

[29:53] In 1900, one of Hilbert's problems was, essentially, to establish the proper foundations of mathematical physics: the problem of the mathematical foundations of mathematical physics. That problem is, of course, still with us — if it ever was a well-defined problem. And, course, in 1900 people didn't know about special relativity, or quantum mechanics, about general relativity, or about all the other things that happened since. So, certainly, it was premature in 1900 to think about, seriously, establishing proper foundations of physics.

[30:24] The question is, "Is the year 2000 a better time?" Well, we don't know — but we can try. A lot has been learnt, and, if so, at what level? Of course, we don't need to try to solve the foundation for everything. We might take the part that we understand well, and try to find the foundation for that. So that's the question we want to pose.

[30:46] Well, first of all, let me give you a quick five-minute review of his physics. Five minutes is fortunate because it's about long enough to cover how much I understand! First of all, we go back to classical physics: Maxwell's equations describing the electromagnetic field, the great theory of the 19th century which, of course, underpins all subsequent physics along way.

[31:15] Then, of course, after the classical equations of magnetic Maxwell's equations, we move on into this century and the discovery of quantum mechanics. The way particles are treated quantum mechanically is a very different approach, of course, from classical theory. From quantum mechanics, we move on to quantum field theory, where you treat not only particles by quantum methods, but also fields themselves. In quantum field theory, the notions of particles and forces all get unified in a very mysterious way. So, quantum field theory is the is a theory which combines fields and quantum mechanics.

[31:53] Quantum electrodynamics is the first attempt to unify things together satisfactorily, incorporating also electromagnetic field and interaction with electrons. Quantum electrodynamics, which developed in the 1950s, is a remarkable theory. It's a very precise theory that gives magnificent confirmation between experiment and theory to very, very high accuracy and these this theory is given by perturbation expansions higher order terms and expansion and there are very complicated rules how you carry this out I call these rules because as a mathematician I can't call it a theory because there isn't a well-defined theory.

[32:35] But physicists use the rules. They don't bother with the fact that there's no theory, and they get very good answers. And it's very hard to argue with something which gives you a rule about computing something which agrees with experimental results to ten figures of accuracy. But there is no rigorous mathematical theory, which is an embarrassment for mathematicians!

[32:56] Now, going beyond quantum electrodynamics, we come into the 1950s, where the Yang-Mills equations were introduced. The Yang-Mills equations are, very crudely speaking, a matrix version of Maxwell's equations. You take eight Maxwell's equations, looked at it in the right way, and you generalize them to matrices. And then, you get a new set of equations called the Yang-Mills equations, and then you treat all the previous theory with an extra complication.

[33:22] You want to do quantum field theory and so on in this situation because these equations are meant to describe not just the electromagnetic field — not just those forces — but also the forces involved in the small-scale structure of matter: the weak forces, and the strong forces. So, the forces involved should be encapsulated in these kinds of enlarged Maxwell's equations.

[33:44] The fundamental formula there, which is the one that links the potential to the field, is the one that says F is the differential of A: $F = dA + A \wedge A$. That would be the formula in the beginning case, but when you have the matrix case, you have a combinator — a bracket — which is non-linear. That's the Yang-Mills complication: Generalizing Maxwell's equations. So, the Yang-Mills equations are non-linear differential equations, whereas Maxwell's equations are linear. And then, of course, you have to build on top of all of that the quantum field theory story.

[34:19] Now, these gauge theories, as they're called, the Yang-Mills theories, are, at present day, the theories accepted by all physicists as the basis for describing the structure of matter — all particles and forces, all forces except for gravitation. QCD — physicists are very good at using short, snappy titles — QED is quantum electrodynamics, in case you didn't know that, and QCD is called quantum chromodynamics, which describes the behavior of quarks.

Difficulties with the Yang-Mills equations

[34:50] Well, that was a quick summary of physics for you. Now, the problems we're going to concern [ourselves] with now is, "How does this develop a quantum field theory based on Yang-Mills equations?" This is a nonlinear equation — much more difficult than quantum electrodynamics — but of course, even quantum electrodynamics wasn't a treat, so you might think this is worse. If we couldn't solve the easy case, how can we solve the harder case?

[35:17] Well, there are reasons to make you believe this is a better theory. There are some good features of the Yang-Mills theory which are not present in the simpler case. There is what is called asymptotic freedom, which tells you about what happens in the very, very small-scale regions. There is also what's called renormalizability: fundamental properties [that have] to do with scale changing. And these are features of the Yang-Mills theory, which are not present in the linear theories. In some sense, the linear theories are too naïve. So, this gives you some hope that this theory will behave better.

[35:50] Now, before going on to describe the problem, let me digress a moment.

[35:54] One of the remarkable things — and this I know more about than the physics — over the last 25 years has been the impact on mathematics of these new ideas in physics: The Yang-Mills equations, the gauge theories, and so on. This impact of physics on mathematics has been truly one of the most remarkable developments of the past 25 years. It's had an enormous range of applications, [with] examples in geometry, topology, algebra, and a whole range of areas — I mentioned just a few. The invariance of knots I mentioned, [from] Vaughan Jones, came out of the story. So does the four-dimensional theory of Donaldson — [it] also comes out of this story. So does one called quantum groups, [which] evolved from here, used very much by group theorists — and I mean I mean real group theory, people [who] do things in characteristic p, and things like that, finite groups. Quantum cohomology is a new notion which emerged, obviously from the terminology, as a kind of hybrid between cohomology theory and quantum ideas, and is a very important application to very classical questions, such as the counting of curves in algebraic geometry. These are a vast range of new ideas come in, with fantastic consequences, to mathematics. So, as a mathematician, I'm very interested in these ideas of physics coming in.

[37:10] One of the drawbacks of using physical idea is that the physics is not rigorously proved. All the ideas you get come incomplete, and therefore the mathematician has to work hard to find some kind of proof of the result that is motivated by the physics. And there's a big industry. One of the biggest industries of the last 20 years is trying to make mathematical proofs of what comes out of the physics. Of course, if you could prove the physics was rigorous, then you would save yourself a lot of work, because all this mathematics would be rigorous, too. In some sense, that's part of the motivation here.

The Yang-Mills Positive Mass Gap Challenge

[37:43] So, Problem 3 of the Clay Institute list is essentially to establish Yang-Mills theory as a rigorous quantum field theory.

[37:52] Now, I have no time to go into the technicalities of what this means, but the important thing is to show what's called the *existence of a positive mass gap*. But the Hamiltonian of the theory's first eigenvalue should be strictly positive. This is [thought] to be true, physicists believe it. The question is to establish a rigorous foundation for the theory, with this as a consequence.

[38:16] Of course this, then, would provide the mathematical basis for the real physics of the world. It would also pass on, by justification, all the [resulting] beautiful applications, to mathematics. So, both physicists and mathematicians would like to know whether this can be done. This is the problem.

[38:30] Now, this is, of course, a *very* difficult problem because it has to do with *infinite* dimensional analysis, because you're dealing with a quantum field theory. It involves analysis, and certainly a lot of topology and algebra, because these spaces are very complicated.

[38:45] This is, I would think the *hardest* of the problems that the Clay Institute has put forward. In some sense, it is the most recent, [though] Yang-Mills equations go back to 1950s. But these attempts to do this are more recent still. [38:59]

Axiomatic quantum field theory

[39:00] Now, I should say a word or two about rigorous attempts to quantum field theory in general. This goes back to this is the work of Arthur Wightman, who laid down an axiomatic approach: What is the quantum field theory; mathematically, what does it mean?

[39:12] He drew up a list of axioms which are *plausible*, and what you would *mean*. And then the question is, can you actually *construct* mathematical objects that satisfy these axioms that will be the physical models you want? That program was then carried out to considerable extent by Arthur Jaffe and [his] colleague [James] Glimm, in dimensions two and three — first in dimension two, and then, with *enormous* efforts, in dimension three. But then they stopped short of dimension four.

[39:40] As we know from geometry and other contexts, the changes in dimension are by no means trivial. You don't just add one and carry on. Enormous new phenomena happen. Things are quite different. The equations for dimension four, for example, are conformally invariant. They're not so in dimension three and two. So, the change in dimensional four is profound, and therefore we don't, at all, know how to carry that out.

[40:04] So that's the big [challenge]. Dimension four is a very special dimension, and the work of Simon Donaldson — [whom] I mentioned in pure geometry, which emerges out of the physics, is an indication. because the things that we discover there are unique to dimension four.

The Future of the Yang-Mills Problem

[40:20] What is the future of this problem in physics? What could happen? This is now my personal guess — or, if you like, hints, suggestions.

Well, you can try a direct analytical assault. If you're a young man, and brave, and you're looking a long way into the future, you can just get out your hammer and chisel and crack away. First, you read [*Quantum Physics: A Functional Integral Point of View*] [by James] Glimm and [Arthur] Jaffe, and then you start on dimension four.

[40:47] Or, you might think that you want to be different. You might find this is basically [a need to] get a better understanding of the formal structure of quantum theories of the Yang-Mills type. These Yang-Mills theories have a very elaborate mathematical formal structure, which incorporates very mysterious dualities of various kinds. It can call for things called supersymmetry, and many applications to mathematics incorporate all these. So here, there is a very elaborate structure, and it is just possible that if we get a better understanding of that structure, then that might give us a better way of trying to lay the foundations.

[41:25] For example, [for] a problem that can be described in two different ways by a duality, the two dual pictures might look quite different. One of them might be tractable, and the other one might be intractable. So it's certainly not at all unreasonable to think this might give you a way of starting — [that is,] if you understand better the formal apparatus. The formal apparatus here is very sophisticated

[41:45] You could, of course, sit back and wait — await development in string theory. String theory — and the latest things that follow string theory, called M-theory, and every year there is a new theory — develop at an enormous rate. The theoretical physics community here is tremendously active. There are very beautiful things happening every week. New results come out, and many of them will have mathematical overtones. It could be that, if we wait a few years, there'll be such a totally new picture emerging from string theory that we'll get a better idea of how to go about attacking the problem. So, that's not an unreasonable expectation.

[42:20] You see, trying to raise foundations for something for physics is like an architect trying to lay the foundation for a building that's rapidly going up. Which you should do first is not quite clear. Do you want you want to see the design, or do you want to lay the foundations? Maybe we want the design first.

[42:40] Then, of course, you might also sit back and say, "Well, quantum electrodynamics was very hard, because we didn't incorporate extra things like protons and neutrons; we only worried about electrons." Maybe even that isn't enough. Perhaps we should incorporate gravity; then we'll have all the forces under control. Perhaps, at that stage, the fundamental theory will be even easier. People in string theory aim here, and this is the ultimate theory — and perhaps the ultimate theory will be easier than the transitional series.

[43:07] But who knows? The big challenge for mathematicians and physicists of the 21st century is to say, really, "Make progress along this program by whatever method you can." Good luck to you!