# INTRODUCTION
## to the book *Computable and Uncomputable*

1. *An algorithm* is a text that in an appropriate environment can lead to a well-determined sequence of events. An enzyme that catalyzes a specific chemical reaction, a field manual, or computer code can be considered as examples of algorithms in the wide sense.

Mathematics furnishes computational algorithms and, more generally, data processing algorithms, which belong to the basic tools of sciences. Moreover, mathematics generates and studies specialized languages used to encode algorithms, to implement them, to describe input and output data, and to design physical devices implementing algorithms. Finally, mathematics produces *theoretical models* of all these notions.

This book is dedicated to some of these theoretical models, specifically to the mathematical theory of algorithmic computability. It is a natural sequel to the book *Provable and Unprovable* (Moscow, "Soviet Radio", 1979), but for the most part can be read independently.

2. The most economical and basic, but extremely universal, model of algorithms assumes that an algorithm prescribes a way to calculate values of a function which is defined on a subset of positive integers $\mathbf{Z}^+$ and which takes positive integer values. For the same function $y = f(x)$ there are many ways to calculate it. It turns out that one may restrict oneself to the following class of calculations, producing all computable functions. Namely, starting with any algorithmic prescription for calculating $f$, one can process it into a polynomial with integral coefficients

$$P_f(x, y; t_1, \dots, t_n)$$

such that $b = f(a)$ if and only if there exist integers $t_1^0, \dots, t_n^0 \in \mathbf{Z}$ satisfying

$$P_f(a, b; t_1^0, \dots, t_n^0) = 0.$$

Knowing $P_f$ and $a$ (in the domain of definition of $f$) we can then determine $b = f(a)$ simply by systematically checking values of $P_f$ on vectors $(a, b; t_1^0, \dots, t_n^0)$.

This is the main result of the first two chapters of this book. A long way leads to it. The first part of the way is devoted to a thorough analysis of the idea of deterministic calculation. This analysis results in postulating that such a calculation can be represented as a sequence of elementary steps, taken from a finite list which is compiled once and fixed forever. The second postulate is that functions computable by an iteration of these elementary steps exhaust the totality of functions deterministically computable in any other imaginable way.

This last statement is essentially an almost physical law which cannot be proved mathematically, but had passed a host of tests and "experimental" verifications.

When Turing, Church, Post, Markov, Kleene, Kolmogorov each suggested and studied their own versions of algorithmic computability, it invariably turned out that these versions were equivalent to the theory *of recursive functions*, which we define and elaborate on in the first Chapter.

The second part of the way towards the result stated above consists of a mathematical treatment of recursive functions with the help of elementary but highly nontrivial arguments of number theory. The initial ideas were introduced in the works of Gödel, Davis, Putnam, J. Robinson, and the decisive final step is due to Yu. Matiyasevich.

3. Any actual computation consists in processing *notations* of numbers, say, in binary form. Before such input data are processed into output data, a computer starts with translating a program, a text written in a programming language, into a machine code. The translator doing this is an algorithm processing symbolic information. Therefore, a mathematical theory of computability based upon recursive functions must include a model of connections between numbers and texts. Chapter IV elaborates one such model, *Gödel's numeration.* Its main result states that all elementary operations on texts that can be used in algorithmic processing of symbolic information become recursive functions after any natural numeration of texts. Hence processing of symbolic information is equivalent to the numerical computation as well.

This result is crucially important because it allows one to unite linguistic and metalinguistic tools in one "computational universe." If both arguments and values of computable functions might be texts then any text describing an algorithm (a metalinguistic tool) can become an input of another algorithm. Moreover, one can imagine an algorithm generating all texts that are descriptions of algorithms.

A study of such universal constructions and the emerging effects of self-referentiality led to the discovery of algorithmically undecidable mathematical problems and unprovable theorems in formal languages.

We prove in Chapter V a very general Gödel's theorem on the incompleteness of formal arithmetic, and dedicate Chapter VI to a subtle undecidable problem of group theory.

Matiyasevich's theorem proved in Chapter II also immediately establishes undecidability of one of the celebrated Hilbert's problems.

4. The very existence of algorithmically unsolvable problems and formally unprovable truths constituted the first fundamental discovery of the theory of computability after the construction of the foundations of this theory. New results of this type continue to appear and to incite considerable interest: cf. e.g., §6 of Chapter V, where we state the strong Ramsey theorem, a rather simple combinatorial statement whose independence from axioms of arithmetic was discovered only in 1977.

But the main trends of the computability theory are related to and motivated by its wider applied, mathematical, and even scientific aspects.

A broad field of applications is known as computer science. It deals with creation of efficient (in terms of time and memory constraints) algorithms for the solution of concrete classes of computational problems. It is engaged as well in devising programming languages, compilers, and general studies of "linguistics of computation." This emerging field, theoretical programming, must work on the brink of the abyss of uncomputability, but its main concern is still how to do well what can be done in principle. The general scope and volume of our book did not allow us to delve into this vast and important domain. The theory of recursive functions only delineates its distant boundaries.

A second direction unites various research programs connecting computability theory with more mainstream mathematical structures. It is well known, for example, that combining the notions of a group structure and of a differentiable manifold one arrives at Lie groups. In the same vein, one can strengthen many mathematical definitions by imposing computability (or more general versions of constructibility) conditions upon the constituent objects and operations. This produces new versions of traditional theories. Constructive fragments of calculus and fragments of other theories are being devised and tested. Such activities are often strongly motivated by a philosophical or simply emotional bias going back to the ancient concerns about "foundations of mathematics." One can hope that these concerns will recede and that the purely mathematical content of such constructions will move to the foreground. Constructive mathematics is not called to replace the classical one, but to become its full-fledged part. In particular, descriptions of recursive structures in set-theoretic terms, along the lines of Matiyasevich (Chapter II) and Higman (Chapter VI), will hopefully bring many more remarkable insights.

An example of a much less straightforward connection of recursivity with classical mathematics is furnished by the deep ideas of A. Kolmogorov in the complexity theory and probability theory. We discuss some of these ideas in Chapter III. Formally speaking, one of the problems solved by Kolmogorov and his students consists of finding a precise definition of random sequences. But at a deeper level, Kolmogorov managed to transform into a well developed theory a vague intuitive feeling that a hidden order in large structures (as opposed to their randomness and chaotic behavior) can only be explicated in the process of algorithmic interaction with these structures which must be far more sophisticated than straighforward prescriptions of statistical counting (cf. §3 of Chapter III).

Therefore, Kolmogorov's theory can also be construed as a contribution of the computability theory to the sciences in general, where the mathematical theory of algorithms is seen as a formalized model of algorithmic processes in the wide sense. As examples of such processes, one can mention translation of texts from one natural language to another one, or else the translation of genotype into phenotype.

Looking at algorithms from this vantage point, one can discover unexpected analogies and problems worth developing. I will sketch below two circles of ideas involving linguistics and physics respectively.

5. *Language* in the wide sense of the word refers to the structure of control and information processes in complex systems. Extending the usage accepted in the linguistics of natural languages, one can refer to concrete fragments of such processes as acts of speech, fixed as texts. The relation between *text and its meaning* should be compared to the relation between *a program and its output*, or *a program and its implementation* rather than the relation between a snapshot of reality and reality itself.

Many years of work on the problems of automatic (or machine) translation led to the crystallization of an important theoretical framework for the description of natural languages: the Meaning-Text theory (cf. I. A. Melchuk, *Outline of a Theory of Meaning-Text Linguistic Models* (in Russian), Moscow, Nauka, 1974). In the framework of this theory, a language is viewed as a many-to-many correspondence between two infinite sets, that of "texts" and that of "meanings". The first set consists of texts in a natural language, whereas the second set consists of texts in an artificial "semantic" language that must be devised. The correspondence in question provides each natural language text with the set of its possible meanings, and each meaning with the set of its correct expressions in the natural language. Linguistics in this sense is the theory of translation, "Meaning $\Leftrightarrow$ Text". The translation passes through a sequence of intermediate *representation levels* of the language.

Each level refers to a description by means of formal languages which must be specially designed. The text itself is first given by a combinatorial object which is called *a Surface Phonological, or Phonetic* representation, whereas a meaning is given by a *Semantic* representation. Successive intermediate levels, ordered from Text to Meaning, are: Deep Phonological, Surface and Deep Morphological, Surface and Deep Syntactic ones.

According to an early summary, "The Meaning-Text model is thus a tool that allows one to describe the production of a wide array of texts in a natural language. This makes it a promising concept for computer implementation and utilization in various systems of automatic processing of text information."

The development of a detailed model of a natural language is a long and labor-intensive endeavour. The analysis of even simple sentences leads to fairly complicated structures on the level of Semantic representation: cf. Figure 1 (next page). The study of semantics involves problems of various nature: semantic analysis as such, synthesis of a formal language capable to express results of such an analysis, design of translating and paraphrasing algorithms. It is not clear what maximal degree of formal description can be attained and what could be the nature of the "un-formalizable residue": is it just an embodiment of random and capricious contingencies which occurred in the history of a natural language, or could there be some deeper roots for its existence?

We can try to shed some light upon this question by considering reduced and simplified versions of the Meaning-Text theory, "models" of this Model. It is convenient to start with a domain of meanings restricted to such a degree that its semantic representation could have as simple a structure as possible. Let us choose for such a domain "natural numbers", with semantic representation of them by sequences of dashes: |, ||, |||, |||| ... . The history of the representation of this domain in natural languages is a part of descriptive linguistics, which simultaneously provides valuable evidence on the early stages of mathematical thought (philogeny).

We learn that a proto-mathematical period is reflected in the following common traits of the counting systems in natural languages.

Some languages assign names only to a few small numbers with all the rest being heaped into one word, "many". In Duwei (a Papua New Guinea language) there are separate words for "one" and "two"; "three" is called "two and one", "four" respectively "two and two". Then "five" refers to the count on fingers: "lima–ngg" = "hand–my".[1] Counting based upon fingers (and generally body parts including

[1]Since this Introduction has been written and published, a considerable new body of field research became available describing Papuan and other proto-mathematical counting systems in fascinating detail. Moreover, such descriptions (based upon the "ethnomathematics" paradigm) show (un)surprising affinities with the Meaning-Text theory. We can refer an interested reader to Glendon Lean's research reviewed in the paper, "Counting Systems of Papua New Guinea," by J. E. Phythian, www.science.uts.edu.au/msc/Language.pdf

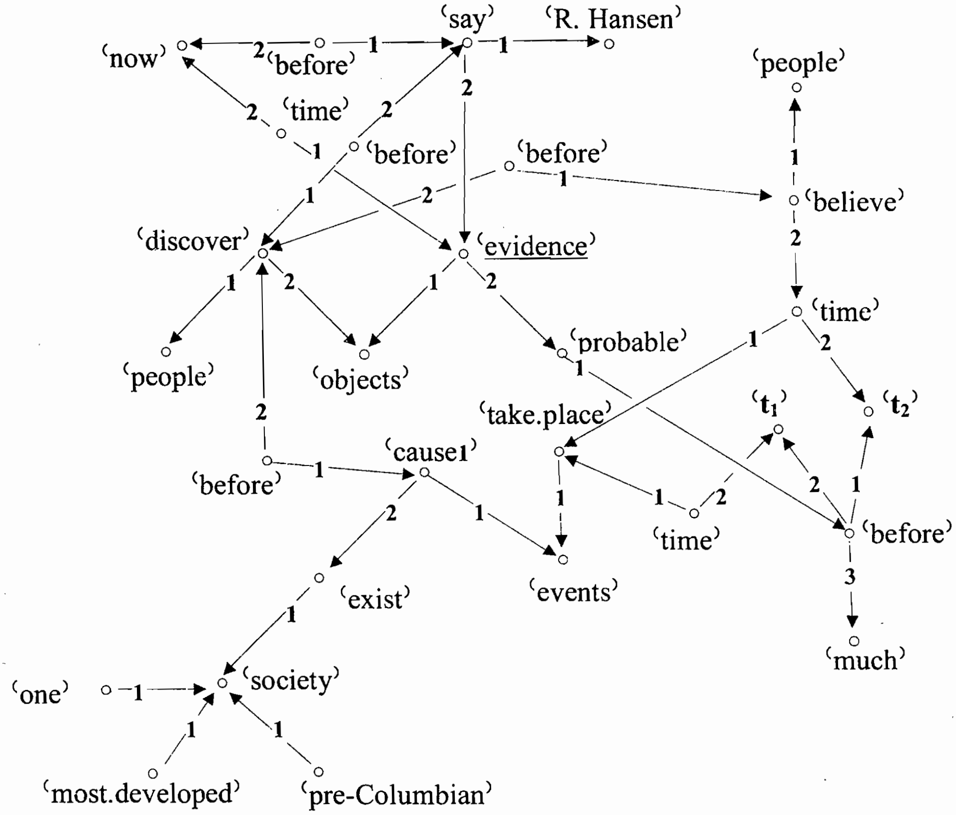


FIGURE 1. This picture is an example of Semantic Representation. It represents the meaning of the sentence *What has been discovered, Richard Hansen said, supplies clear indication that the achievements which had created the most advanced Pre-Columbian society may have taken place much earlier than was previously believed.* I am grateful to Igor Melchuk who supplied this illustration as a replacement for the original example in Russian, and allowed me to reproduce it here.

ears and eyes), accompanied by words for these body parts, continues up to, say, 20. There might exist separate names for certain larger numbers, often with growing lacunae, and then the count stops, for the lack of a systemic pattern of forming further names. Of course, a system like $\{1, \dots, 20, many\}$ can be formalized as a legitimate "small universe" of mathematics. Moreover, a philosophical propaganda for similar universes (as opposed to the classical natural numbers) is elaborated by one of the radical schools in foundations of mathematics ("ultrafinitism" of A. Essenin-Volpin). However, this refined and cultivated regress to the archaic stages of consciousness cannot really convince us to throw away the powerful idea of the potential or even actual infinity of natural numbers.

Some natural languages use different series of numerals for counting things with different properties (long, round, animate, inanimate, etc.) These can be considered as remnants of a long era during which the idea that the same number can be used for counting whatever was forming. Human consciousness took a long time to get accustomed to the idea of putting into one class various sets of the same cardinality, even when this cardinality was finite and small. Applied to infinite sets, this idea became a mathematical tool only after G. Cantor's work. Traces of such archaic counting systems are conserved in many languages, for instance, in modern Chinese: it possesses a unified system of numerals, complemented by a system of counting words used with nouns of various classes like *kuài* ("piece"), *ge* ("thing"), *běn* ("root"), etc.

In many languages, some ordinal numerals have roots different from the respective cardinal numerals: cf. one/first, two/second and unus/primus, duo/secundus in Latin. These examples can be treated as signs of a very early recognition of the idea of order as opposed to the idea of quantity, the idea that crystallized into a mathematical notion remarkably late—only in the 19th–20th centuries (Cantor's "ordinals" vs. "cardinals", Bourbaki's ordered sets).

The first written texts that we have (Babylon, Egypt) already reflect a rich picture of developed mathematical knowledge, in particular, a conception of the language of mathematical notation sufficiently separate from the natural language. The central place in it is taken by a system of notations for numbers and operations on them. The prehistory of the place value systems of modern type is based upon the idea of counting groups (bases) of growing sizes. These bases may be powers of one and the same number (the base of a place value system: 60 in Babylonian mathematics, our decimal and binary systems etc.), but historically other options were used as well. Thus, in Maya's chronology, bases are 1, 18, 360. The traces of the archaic counting with the base 20 are conserved in French names of numerals, as *quatre-vingt-six* ($= 4 \times 20 + 6$.) The number of bases of the respective size is denoted by a special symbol. Initially, this symbol may depend on the respective base (as in ancient Egypt, Greece, Rome). When this symbol becomes universal and the base itself becomes implicitly determined only by its position in a chain of signs (*digits* constituting *a numeral*) designating a number (*the value* of this numeral), a special symbol for zero must be included in the system in order to ensure unique interpretation. Its appearance and acceptance is the result of a long evolution. By the end of the Babylonian tradition, the absence of the base of a given size is already marked in middle and end positions; zero in the initial position indicates that the following numeral refers to the sexagesimal fractions of the unit.

The idea that there should exist a specific numeral whose value is a legitimate number "zero" came into being even later. Apparently, it originated with the Hindu people in India, and through Arabs was appropriated by the European mathematics.

A place value notation already implicitly encodes elementary operations on numbers: to transform a numeral into its value, one has to multiply each digit by the value of its position and then add all the products. Systematic rules for implementing arithmetic operations on numerals in a place value system were described in the IX century by Muhammad bnu Musa al-Khwarizmi in the influential books *Hisab al-jabr wa al-muqabala* (a compendium of *Calculus by Completion and Balancing*) and *On the Calculus of the Hindus*. His name (actually derived from the toponym Khorezm, a Persian town, now in Uzbekistan) was transformed into our word "algorithm", and "al–jabr" from the title of his book became a generic word for the western "algebra".

At this turn in the road, counting systems in natural languages gradually and paradoxically change their semantics. "Nineteen hundred eighty-four" is the name of the decimal numeral 1984 rather than the name of the value of this numeral. Hence words turn into secondary signs, denoting not an object or an idea but another sign. (Imagine the specific psychological difficulty involved in reading binary numerals: the value of 1000 in binary notation is eight, but the word "eight" as a substitute for 1000 sounds weird because this word became the name of a name, 8).

We will end now our brief discussion of counting systems of natural languages and instead try to imagine the most robust characteristic traits of any symbolic counting system, not necessarily a place value one. The minimal requirements for such a system are the following ones:

(i) Numerals must be finite texts.

(ii) The function *a numeral* $\mapsto$ *its value* must be computable, that is, describable by an algorithm.

But if these two conditions were sufficient, it would be pointless to look for something better than just $|, ||, |||, \ldots$. In fact, place value systems became so widespread because they implement a fundamental mathematical discovery: *any integer $N$ can be written using no more than about* $\log N$ *letters of a finite alphabet.* Therefore, $N$ sticks or notches is a highly wasteful notation for $N$.

Al-Khwarizmi's rules show the second great advantage of the place value systems: the algorithms of basic calculations with place value numerals allow short descriptions and reasonably short implementation.[2]

Hence, we expect at least:

(iii) Numerals must be short.

This remark leads to the following idea: let us consider *any* computable function taking all positive integer values $f$ as a model of numerals (recall that texts can be replaced by their Gödel numbers). Temporarily disregarding computability of operations on numerals, let us focus on the idea of *economy*: we are interested in such functions $f$ that any number $N$ has a name as short as possible, where a name of $N$ is any $n$ such that $f(n) = N$. For example, the binary length of "$10^{10^{\cdots^{10}}}$ (1000 exponents)" is very large, but we managed to express it using only a few letters.

A. N. Kolmogorov has proved that such optimal functions do exist and can be described explicitly. Each such $f$ calls $N$ by a name as short as possible within a bounded error in the following sense: the shortest name of $N$ with respect to any other function $g$ can be shorter than the optimal one only by a number of bits restricted by a constant depending on $f$ and $g$, but not on $N$.

However, having reached this optimal encoding, we have to pay a price.

Any optimal counting system has the following properties:

a) *Each number has infinitely many names.*

b) *Not every number is a name: each Kolmogorov optimal function is only partially recursive, and cannot be extended to a general recursive one.*

c) *In order to reconstruct a number knowing its name, we must apply a complex algorithm: optimal functions are constructed using universal computable functions, which in a sense are as complex as computable functions can be.*

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[2]In order to appreciate this property, the reader is invited to consider finite continued fractions as notations for rational numbers and to devise algorithms of their addition and multiplication.

d) *The problem of finding the shortest name of a number, that is, the function* $\nu(N) = \min\{\, n \mid f(n) = N \,\}$, *is not algorithmically solvable:* $\nu(N)$ *is not a recursive function. To put it more suggestively, analyzing a large number (or generally a long text) in order to discover its hidden structure, which would allow us to compress it as efficiently as possible is a creative problem that cannot be left to a mechanical device.*

Let us now compare this list of properties of an optimal counting system with the following properties of natural languages:

A. *Abundance of synonymous expressions: any meaning can be expressed by a great variety of texts in a natural language.*

(According to I. Melchuk's calculations, the sentence represented in Figure 1 admits about fifteen million paraphrases.)

B. *Openness of language: at each given moment, not all grammatically correct texts have a meaning.*

(This short statement deserves a more extended discussion. The model "Meaning-Text" postulates that each correct text can be translated into a text in the language of meanings, but the latter can be "meaningless" in the informal sense of this word: thus, our observation is simply shifted to another level).

This openness of a natural language is an extremely important reserve allowing creative use of the language not only in poetry and philosophy, but in sciences as well. To express a newly emerged meaning one can use a text which was "meaningless" before, such as "wave of probability", or more down-to-earth, "packet/carton of milk". Even more interesting are occurrences of the birth of new meanings in poetic or scientific metaphors (Adam Smith's "invisible hand of market", Feynman's "path integrals").

C. *The translation "Text ⇒ Meaning" requires implementation of a system of elaborate algorithms making explicit the high structural complexity of expressions in a natural language.*

D. *In all existing studies, the translation "Meaning ⇒ Text" turns out to be an even more challenging undertaking.*

Comparing properties a) – c) to A. – C., we observe their remarkable parallelism. This allows us to conjecture that some characteristic traits of natural languages, usually treated as historic contingency, on a deeper level are connected with economy requirements: the existence of optimally short expressions of complex meanings. The abundance of paraphrases and "meaningless" texts superficially contradicts this conjecture, but if our model/metaphor is accepted, this abundance paradoxically turns out to be an unavoidable consequence of optimality.

6. In post-Newtonian classical physics, the idea of determinacy is usually expressed in the form of the following principle: the space-time evolution of an isolated physical system is determined by *differential equations* ("laws of nature") and *initial/boundary conditions.* The same principle is valid in the (Hamiltonian version of) quantum physics: time evolution of quantum observables is governed by differential equations and initial conditions. The probabilistic aspect of quantum physics becomes essential for the description of interactions, in particular, with a measuring device, but not for the description of an isolated system.

A computation can be viewed as an alternative expression of the idea of determinacy. In many respects, it is parallel to the classical one: "laws" are encoded in the structure of the processor, memory, and input/output devices, whereas initial/boundary conditions are programs. A computation is a sequence of elementary steps; in particular, the simplest changes in the state of the memory device (imagine erasing/writing one bit on the tape of a Turing machine) can be correlated with the idea of a differential in calculus. When we numerically solve, say, a heat equation, we imitate in a pretty straightforward way the continuous determinacy replacing it by the discrete one, but generally the comparison of the evolution of a physical system with a solution of its numerical model can be highly non-trivial.

Molecular biology furnishes examples of the behaviour of natural (not engineered by humans) systems which we have to describe in terms initially devised for discrete automata. Figure 2 (next page) is a scheme of RNA translation and protein synthesis.[3] It does look somewhat like a drawing of a Turing machine copying information from one tape to another.

A classical continuously evolving system governed by differential equations can imitate a discrete automaton only if its phase space is extremely elaborate: it must include many stability domains, or attractors, separated by low energy barriers. The input of a program creates a labyrinthine system of passages in these barriers creating a path for a trajectory that approximates the discrete process of computation. As a physical system, a computer must be extremely unstable because a change in one bit of input generally leads to a totally different computation. But the computation itself as a physical evolution must be very stable: jumping over a closed barrier as a result of physical fluctuations must be highly improbable. It is well known that these requirements (combined with inherent speed restrictions and the growth of dissipated energy with the growth of complexity) doomed the development of mechanical computers. Nevertheless, we believe that "genetic automata" can be described in such mechanical terms. One of the well-known problems to which such a description leads is a picture of DNA replication. Replication of the double helix of a bacterial chromosome involves the uncoiling of about 300,000 turns accompanied by an intricate set of highly specific chemical reactions.

Perhaps, for a better understanding of this phenomenon, we need a mathematical theory of quantum automata. Such a theory would provide us with mathematical models of deterministic processes with quite unusual properties. One reason for this is that the quantum state space has far greater capacity than the classical one: for a classical system with $N$ states, its quantum version allowing superposition (entanglement) accommodates $c^N$ states. When we join two classical systems, their number of states $N_1$ and $N_2$ are multiplied, and in the quantum case we get exponential growth $c^{N_1 N_2}$.

These crude estimates show that the quantum behavior of the system might be much more complex than its classical simulation. In particular, since there is no unique decomposition of a quantum system into its constituent parts, a state of the quantum automaton can be considered in many ways as a state of various virtual classical automata. Cf. the following instructive comment at the end of the article by R. P. Poplavskii, "Thermodynamical models of information processing" (in Russian), *Uspekhi Fizicheskikh Nauk*, **115:3** (1975), 465–501: "The quantum-mechanical computation of one molecule of methane requires $10^{42}$ grid

[3]The figure presented here was redrawn for this edition by M. Gelfand and O. Khleborodova, to whom I owe my deepest graptitude.

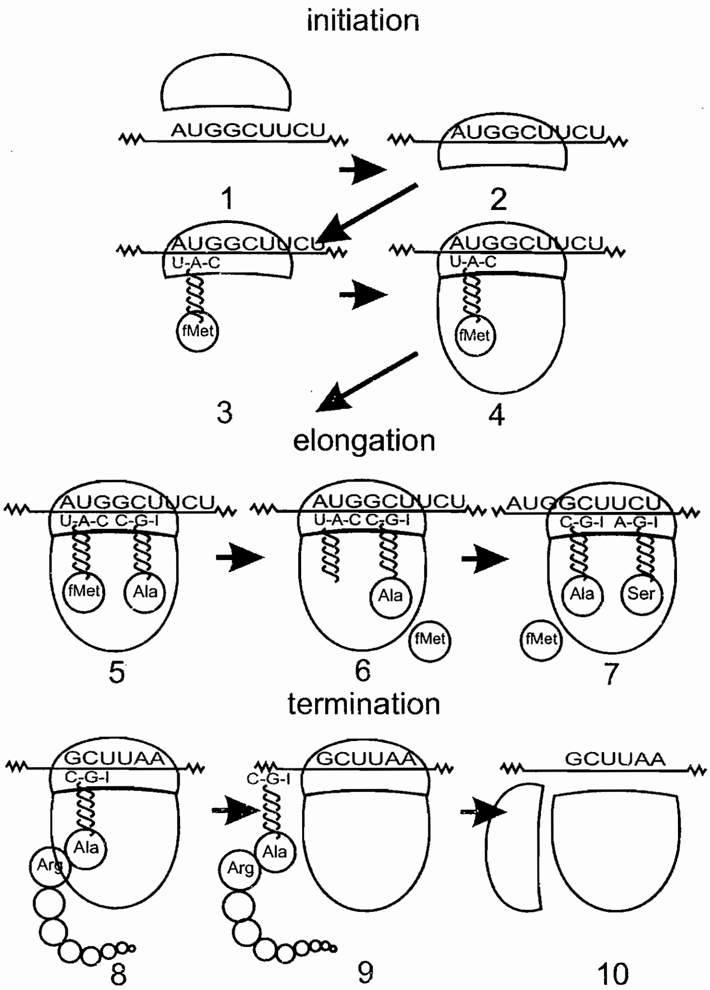


FIGURE 2

points. Assuming that at each point we have to perform only 10 elementary operations, and that the computation is performed at the extremely low temperature $T = 3 \cdot 10^{-3} K$, we would still have to use all the energy produced on Earth during the last century."

The first difficulty we must overcome is the choice of the correct balance between the mathematical and the physical principles. The quantum automaton has to be an abstract one: its mathematical model must appeal only to the general principles of quantum physics, without prescribing a physical implementation. Then the model of evolution is the unitary rotation in a finite dimensional Hilbert space, and the decomposition of the system into its virtual parts corresponds to the tensor product decomposition of the state space ("quantum entanglement"). Somewhere in this picture we must accommodate interaction, which is described by density matrices and probabilities.