distinction between geometric and kinematic shapes; a statement of geometric content concerns the kinematic or geometric shape, depending on whether it is related to a reference system S or not.

§ 3. Coordinate-Time Transformation.

Let S and S' be equivalent reference systems, i.e., these systems have uniform scales of equal length and clocks that run at the same rate if these objects are compared to each other while in a state of relative rest. It is then obvious that every natural law that applies with respect to S also applies in exactly the same form with respect to S' if S and S' are at rest relative to one another. The principle of relativity requires this perfect agreement even in the case that S' is in uniform translational motion relative to S. In particular, the speed of light in a vacuum must be the same number in relation to both reference systems.

Let a point event be determined relative to S by the variables x, y, z, t, and relative to S' by the variables x', y', z', t', where S and S' are free of acceleration and moving relative to each other. We now seek the equations that transform between the former and the latter variables.

We can immediately say that these equations must be linear with respect to the variables mentioned, because the homogeneity properties of space and time require this. In particular, it follows that the coordinate planes of S' — related to the reference system S — are uniformly moving planes; however, these planes will generally not be perpendicular to one another. However, if we choose the position of the x'-axis so that the latter — relative to S — has the same direction as the translational movement of S' relative to S, then for reasons of symmetry it follows that the coordinate planes of S' relative to S must be perpendicular to each other. Thus we can and will choose the positions of the two coordinate systems in such a way that the x-axis of S and the x'-axis of S' constantly coincide and that the y'-axis of S' related to S is parallel to the y-axis of S. Furthermore, we want to choose as the starting point of time in both systems the moment at which the coordinate starting points coincide; then the linear transformation equations sought are homogeneous.

From the now-known position of the coordinate planes of S' relative to S, we immediately conclude that the following pairs of equations are equivalent:

$$x' = 0 \quad \text{and} \quad x - vt = 0$$
$$y' = 0 \quad \text{and} \quad y = 0$$
$$z' = 0 \quad \text{and} \quad z = 0$$

Three of the transformation equations we are looking for are of the form:

$$x' = a(x - vt)$$
$$y' = by$$
$$z' = cz\,.$$

Since the speed of propagation of light in empty space is the same c with respect to both reference systems, the two equations:

$$x^2 + y^2 + z^2 = c^2t^2$$

and

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

must be equivalent. From this and from the expressions just found for x', y', z', one concludes after simple calculation that the transformation equations sought must be of the form:

$$t' = \varphi(v) \cdot \beta \cdot \left(t - \frac{v}{c^2}x\right)$$
$$x' = \varphi(v) \cdot \beta \cdot (x - vt)$$
$$y' = \varphi(v) \cdot y$$
$$z' = \varphi(v) \cdot z\,.$$

where

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,.$$

We now want to determine the still-undetermined function of v. Let us introduce a third reference system S'', equivalent to S and S', which moves relative to S' with the velocity $-v$ and is oriented relative to S' in the same way that S' is oriented relative to S. By twice applying the equations just derived we obtain:

$$t'' = \varphi(v) \cdot \varphi(-v) \cdot t$$
$$x'' = \varphi(v) \cdot \varphi(-v) \cdot x$$
$$y'' = \varphi(v) \cdot \varphi(-v) \cdot y$$
$$z'' = \varphi(v) \cdot \varphi(-v) \cdot z\ .$$

Since the coordinate starting points of S and S'' constantly coincide, the axes are oriented in the same way, and the systems are "equivalent", this substitution is identical [1]). Thus

$$\varphi(v) \cdot \varphi(-v) = 1 .$$

Furthermore, since the relationship between y and y' cannot depend on the sign of v,

$$\varphi(v) = \varphi(-v) .$$

So [2]) $\varphi(v) = 1$, and the transformation equations are

$$\left.\begin{aligned} t' &= \beta\left(t - \frac{v}{c^2}x\right) \\ x' &= \beta(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

If you solve the equations (1) for x, y, z, t, you get the same equations, only with: "primed" quantities replaced by the "unprimed" quantities of the same name, and vice versa; and v replaced by $-v$. This also follows directly from the principle of relativity, and from the consideration that S carries out a parallel translation relative to S' in the direction of the X' axis with the speed $-v$.

In general, according to the principle of relativity, one gets a new correct relationship from every correct relationship between "primed" (defined with reference to S') and "unprimed" (defined with reference to S) quantities, or between quantities of only one of these system types, if one replaces the unprimed characters by the corresponding primed characters, and vice versa, and v by $-v$.

§ 4. Conclusions from the transformation equations concerning rigid bodies and clocks.

1. A body is at rest relative to S'. Let x'_1, y'_1, z'_1 and x'_2, y'_2, z'_2 be the coordinates of two material points related to S'. Between the coordinates x_1, y_1, z_1 and x_2, y_2, x_2 of these points

1) This conclusion is based on the physical assumption that the length of a ruler and the speed of a clock do not suffer any permanent change as a result of these objects being set in motion and brought to rest again.

2) $\varphi(v) = -1$ obviously is not an option.